

EXACT S-MATRICES WITH
AFFINE QUANTUM GROUP SYMMETRY

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e-mail: delius@mth.kcl.ac.uk***ABSTRACT**

We show how to construct the exact factorized S-matrices of 1+1 dimensional quantum field theories whose symmetry charges generate a quantum affine algebra. Quantum affine Toda theories are examples of such theories. We take into account that the Lorentz spins of the symmetry charges determine the gradation of the quantum affine algebras. This gives the S-matrices a non-rigid pole structure. It depends on a kind of “quantum” dual Coxeter number which will therefore also determine the quantum mass ratios in these theories. As an example we explicitly construct S-matrices with $U_q(c_n^{(1)})$ symmetry.

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1 Introduction

This paper is concerned with the exact determination of the S-matrices of certain 1+1 dimensional quantum field theories.

It is highly desirable to know S-matrices exactly because the complete on-shell information about a quantum field theory is contained in its S-matrix. In general however one is forced to resort to perturbative or otherwise approximative solutions. Many interesting phenomena do not show up in perturbation theory. The study of exact S-matrices in 1+1 dimensions can shed light on such phenomena. We would like to mention two examples: 1) The scalar S-matrices for the fundamental particles in real coupling affine Toda theory [5, 12] display a strong coupling — weak coupling duality which at the same time interchanges a Lie algebra with its dual algebra. 2) In the Sine-Gordon theory the exact breather S-matrix [33] is equal to the S-matrix of the fundamental particle, suggesting that the breather solutions and fundamental particles are just different descriptions of the same object. This has recently been extended to $a_2^{(1)}$ Toda theory [19]. These are exactly the kind of phenomena which have been conjectured to occur in 4-dimensional Yang-Mills theory [20, 28] and have recently been followed up by Seiberg and Witten [32].

The known exact 1+1 dimensional S-matrices are either diagonal or are proportional to a rational or trigonometric R-matrix. Rational R-matrices are intertwiners of representations of Yangian algebras [17] and give the S-matrices of the principal chiral models [2, 29]. They have the feature that they do not depend on any parameter. Trigonometric R-matrices are intertwiners of representations of quantum affine algebras $U_q(\hat{g})$ [24, 17]. These are deformations of the enveloping algebras of affine Kac-Moody algebras [25] and depend on a parameter q . For $\hat{g} = a_n^{(1)}$ these R-matrices give the soliton S-matrices of $a_n^{(1)}$ affine Toda theory [22] (and in particular for $a_1^{(1)}$ give the Sine-Gordon S-matrix). This implies that these theories have a $U_q(a_n^{(1)})$ quantum affine symmetry which had been observed previously by Bernard and LeClair [3]. At q a root of unity the trigonometric R-matrices give the S-matrices of perturbed W-invariant theories [10, 1].

In this paper we study quantum field theories with quantum affine symmetry in general. We show how their S-matrices are obtained from the universal R-matrices of the quantum affine algebras. Even though the theory of quantum affine algebras and their quasitriangular structure given by the universal R-matrices was not originally designed for this purpose, it turns out to be the ideal basis for the construction of solutions to the S-matrix axioms. Indeed the S-matrix axioms of unitarity, crossing symmetry and the bootstrap principle follow directly from the fundamental properties of the universal R-matrices. In addition, one gets factorization, which expresses the multi-particle S-matrices in terms of the two-particle S-matrices, for free. Schematically (we will give details later):

$$\begin{aligned} (\mathcal{S} \otimes 1)R &= R^{-1} \Rightarrow \text{crossing symmetry} \\ (\Delta \otimes 1)R &= R_{13}R_{12} \Rightarrow \text{bootstrap principle} \\ R\Delta &= \Delta^T R \Rightarrow \text{unitarity and factorization} \end{aligned} \tag{1.1}$$

We will show the importance of the Lorentz spins of the symmetry charges. They determine the gradation of the quantum affine algebras and the S-matrices depends crucially on this. The influence which the gradation has on the R-matrices was described by us in [4] and has since been independently used in [1]. We show how the locations of the particle poles of the S-matrices and therefore the particle quantum mass ratios depend on both the deformation parameter q and on the gradation, see eq.(2.51). In particular this overcomes the restriction to unrenormalized mass ratios hitherto observed in exact soliton S-matrices.

We study the requirement of crossing symmetry and find that it places a constraint on the possible gradation. This leads to the formula (2.52) in which a “quantum” dual Coxeter number occurs. This extends the observation in real coupling Toda theory that quantum effects tend to manifest themselves through the replacement of the Coxeter number by a “quantum” Coxeter number [12].

These general results are contained in section 2 and in section 3 we demonstrate the general framework with the example of S-matrices with $U_q(c_n^{(1)})$ quantum affine symmetry.

We will not include the application of our study to quantum affine Toda theories in this paper. It will appear in a future joint publication with Gerard Watts and Nial MacKay.

2 S-matrices with quantum affine symmetry

Before specializing to an example in the next section, we will here study the properties of the S-matrices of any two-dimensional relativistic quantum field theory which has a quantum affine symmetry $U_q(\hat{g})$. We start in section 2.1 by defining what we mean by a quantum affine symmetry. We introduce the two-particle S-matrix in section 2.2 and show how it is expressed through the R-matrix of $U_q(\hat{g})$. We give some relevant information about the R-matrices in section 2.3. In section 2.4 we derive unitarity and crossing symmetry from the properties of $U_q(\hat{g})$. We discuss the S-matrix pole structure in section 2.5. In section 2.6 we will see how the quantum affine symmetry fixes the particle pole locations and thus determines the quantum mass ratios.

2.1 Quantum affine symmetry

We say that a relativistic quantum field theory has a quantum affine symmetry $U_q(\hat{g})$ if the following two properties hold:

1) *The theory possesses quantum conserved charges H_i, X_i^\pm , $i = 0 \cdots r$, which obey the same relations as the Chevalley generators of $U_q(\hat{g})$.* Thus they obey the

commutation relations

$$\begin{aligned}
[H_i, X_j^\pm] &= \pm a_{ij} X_j^\pm, \\
[X_i^+, X_j^-] &= \delta_{ij} \frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_i^{-1}}, \quad q_i \equiv q^{d_i}, \\
[H_i, H_j] &= [X_i^\pm, X_j^\pm] = 0,
\end{aligned} \tag{2.1}$$

and also the quantum Serre relations, which we will not write down here. For background on quantum affine algebras see e.g. [8]. In the above, a_{ij} is the generalized Cartan matrix of an affine Kac-Moody algebra \hat{g} [25] and the d_i are coprime integers such that the matrix $(d_i a_{ij})$ is symmetric. q is a complex parameter which will be related to Planck's constant \hbar and the coupling constant of the field theory.

2) *The conserved charges possess a definite Lorentz spin.* Thus if D denotes the infinitesimal two-dimensional Lorentz generator, then we require that

$$[D, X_i^\pm] = \pm s_i X_i^\pm, \quad [D, H_i] = 0, \quad i = 0, \dots, r. \tag{2.2}$$

$s_i \in \mathbb{R}$ is called the Lorentz spin of X_i^\pm . The fact that the X_i^- have Lorentz spin $-s_i$ and that the H_i have Lorentz spin 0 is required by consistency with the commutation relations (2.1).

The operators $X_i^\pm, H_i, i = 0, \dots, r$, and D together generate the quantum enveloping algebra $U_q(\hat{g})$. D is called the derivation. It is because the Lorentz transformation is integrated into the quantum affine symmetry algebra in this way, that this symmetry gives strong constraints on the form of the S-matrix. We will denote the algebra without the derivation D by $U_q(\hat{g})$.

Let the Lorentz spin of an operator A be denoted by $s(A)$. Then s satisfies $s(AB) = s(A) + s(B)$. Thus $s : U_q(\hat{g}) \rightarrow \mathbb{R}$ is a gradation of $U_q(\hat{g})$. Such a gradation is uniquely fixed by the vector $s = (s_0, \dots, s_r) \in \mathbb{R}^{r+1}$. The most common gradations used in studying affine algebras are the homogeneous gradation which has $s_0 = 1$ and all other $s_i = 0$, and the principal gradation which has all the $s_i = 1$. We will see that interesting physical effects arise from studying more general gradations, in particular gradations which depend on the coupling constant of the field theory.

We now start to consider the consequences which the presence of a quantum affine symmetry has for the theory.

The quantum affine symmetry implies quantum integrability of the theory. Quantum integrability is given when one can find an infinite number of commuting higher spin conserved charges. The infinitely many Casimir operators of $\widetilde{U_q(\hat{g})}$ supply such higher spin conserved charges. They are not the standard local integer spin charges usually considered [30], but they have the same strong implications. For example their conservation guarantees that in a scattering process the set of incoming momenta equals the set of outgoing momenta. We arrange the particles into multiplets under these charges. By a multiplet we mean the collection of all particles with the same mass and the same eigenvalues under all the higher spin Casimir operators.

The multiplets of one-particle states will transform in the finite dimensional irreducible representations of $\widetilde{U_q(\hat{g})}$ uniquely determined by the values of all the higher Casimir operators.

We denote the one-particle states by $|a, \alpha, \theta\rangle$, where a denotes the multiplet, α labels the particle within the multiplet and θ is the rapidity of the particle. The rapidity specifies the energy $E = m \cosh(\theta)$ and the momentum $p = m \sinh(\theta)$ of the particle, m being the mass of the particle. At fixed rapidity the particles in the multiplet a span the space V_a which carries a finite dimensional unitary representation π_a of $\widetilde{U_q(\hat{g})}$. The central charge of the algebra takes the value zero in all finite dimensional representations.[‡] Including the rapidity the one-particle space is $V_a \otimes \mathcal{F}$, where \mathcal{F} is a suitably chosen space of functions of θ .[§] Under a finite Lorentz transformation $L(\lambda) = \exp(\lambda D)$ the rapidity θ is shifted by λ

$$L(\lambda)|a, \alpha, \theta\rangle = |a, \alpha, \theta + \lambda\rangle. \quad (2.3)$$

From this we deduce that $V_a \otimes \mathcal{F}$ carries the following infinite dimensional representation $\pi_{s,a}$ of $U_q(\hat{g})$, where the subscript s denotes the gradation

$$\begin{aligned} \pi_{s,a}(D) &= 1 \otimes \frac{d}{d\theta}, \\ \pi_{s,a}(X_i^\pm) &= \pi_a(X_i^\pm) \otimes e^{\pm s_i \theta}, \\ \pi_{s,a}(H_i) &= \pi_a(H_i) \otimes 1. \end{aligned} \quad (2.4)$$

The appearance of $e^{\pm s_i \theta}$ in $\pi_{s,a}$ is dictated by (2.1). Thus the one-particle states with definite rapidity θ transform under an element $A \in \widetilde{U_q(\hat{g})}$ as

$$|a, \alpha, \theta\rangle \mapsto \pi_{s,a}^{(\theta)}(A)_{\alpha\beta} |a, \beta, \theta\rangle \quad (2.5)$$

where we have defined the family of finite dimensional representations $\pi_{s,a}^{(\theta)}$ of $\widetilde{U_q(\hat{g})}$ by

$$\pi_{s,a}^{(\theta)}(H_i) = \pi_a(H_i), \quad \pi_{s,a}^{(\theta)}(X_i^\pm) = e^{\pm s_i \theta} \pi_a(X_i^\pm). \quad (2.6)$$

We will usually drop the subscript s denoting the gradation if it is clear from the context.

We can also derive the action of the symmetry on asymptotic multi-particle states. We assume that asymptotically, when the particles are far apart, a two-particle state can be represented as a tensor product $|a, \alpha, \theta\rangle \otimes |b, \beta, \theta'\rangle$ of two one-particle states. We choose the ordering of the factors in the tensor product

[‡]For the reader who is wondering how this is consistent with the statement that there are no finite dimensional unitary highest-weight representations of affine Lie algebras: these are not highest-weight representations in the usual sense. For a treatment of finite dimensional representations of quantum affine algebras see [7, 14].

[§]Strictly speaking the one-particle states of definite rapidity do not lie in this space but need to be smeared by test functions as $\int d\theta' f(\theta - \theta') |a, \alpha, \theta\rangle$, but all these details do not need to concern us here.

according to the ordering of the particles in space, i.e. the first is to the left of the second. Consistency with the commutation rules (2.1) implies that the action of the symmetry on such a state is given by the coproduct Δ of $U_q(\hat{g})$ [¶]

$$\begin{aligned}\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, & \Delta(D) &= D \otimes 1 + 1 \otimes D, \\ \Delta(X_i^\pm) &= X_i^\pm \otimes q_i^{H_i/2} + q_i^{-H_i/2} \otimes X_i^\pm.\end{aligned}\tag{2.7}$$

i.e that

$$(|a, \alpha, \theta\rangle \otimes |b, \beta, \theta'\rangle) \mapsto \pi_{ab}^{(\theta\theta')}(A)_{\alpha\alpha', \beta\beta'}(|a, \alpha', \theta\rangle \otimes |b, \beta', \theta'\rangle) \tag{2.8}$$

$$\text{where } \pi_{ab}^{(\theta\theta')}(A) = \left((\pi_a^{(\theta)})_{\alpha\alpha'} \otimes (\pi_b^{(\theta')})_{\beta\beta'} \right) \Delta(A), \tag{2.9}$$

Such a nontrivial action on an asymptotic two-particle state, where the action on the one particle depends on the state of the other particle even though it is very far away, is possible only for nonlocal symmetry charges. The action on n-particle states is

$$\pi_{a_1 \dots a_n}^{(\theta_1 \dots \theta_n)}(A) = (\pi_{a_1}^{(\theta_1)} \otimes \dots \otimes \pi_{a_n}^{(\theta_n)}) \Delta^{n-1}(A), \tag{2.10}$$

where $\Delta^2 = (1 \otimes \Delta)\Delta$, $\Delta^3 = (1 \otimes 1 \otimes \Delta)\Delta^2$, etc.

2.2 The two-particle S-matrices

We now introduce the two-particle S-matrices $S_{ab}(\theta - \theta')$ describing the process depicted in figure 1.

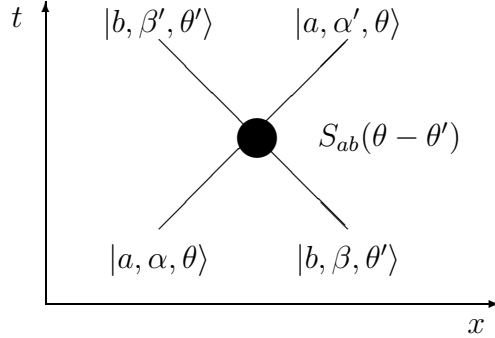


Figure 1: The two-particle scattering process described by $S_{ab}(\theta - \theta')$

Note that due to the integrability of the theory, i.e. due to the conservation of the higher-spin charges, these are the only two-particle processes which are allowed. There can be no change of particle multiplets and no change of rapidities. Only the particles within a multiplet can be converted into each other. Lorentz invariance dictates that S_{ab} depends on $\theta - \theta'$ only. See [33] for a discussion of scattering in integrable quantum field theories.

[¶]The algebra relations (2.1) are invariant under $q \leftrightarrow q^{-1}$, but our choice of the coproduct 2.7, rather than its opposite, fixes q .

The S-matrix $S_{ab}(\theta - \theta')$ gives the mapping of an incoming asymptotic two-particle state into an outgoing asymptotic two-particle state

$$S_{ab}(\theta - \theta') : V_a(\theta) \otimes V_b(\theta') \rightarrow V_b(\theta') \otimes V_a(\theta) \quad (2.11)$$

$$|b, \beta', \theta'\rangle \otimes |a, \alpha', \theta\rangle = (S_{ab}(\theta - \theta'))_{\alpha'\alpha, \beta'\beta} (|a, \alpha, \theta\rangle \otimes |b, \beta, \theta'\rangle) \quad (2.12)$$

The quantum affine symmetry tells us that

$$S_{ab}(\theta - \theta') \pi_{ab}^{(\theta\theta')}(A) = \pi_{ba}^{(\theta'\theta)}(A) S_{ab}(\theta - \theta'), \quad \forall A \in \widetilde{U_q(\hat{g})}. \quad (2.13)$$

This is just saying that, by the definition of a symmetry, it must not make a difference whether we first perform a symmetry transformation and then scatter or first scatter and then perform the symmetry transformation.

According to (2.13) $S_{ab}(\theta - \theta')$ is an intertwiner between the representation $\pi_{ab}^{(\theta\theta')}$ and the representation $\pi_{ba}^{(\theta'\theta)}$. Because these representations are irreducible for generic θ, θ' , such an intertwiner is unique, up to an overall constant. This intertwiner is obtained by evaluating the universal R-matrix of $\widetilde{U_q(\hat{g})}$ in the appropriate representation and gradation

$$\check{R}_{ab}^{(s)}(\theta - \theta') = \sigma_{ab} \left((\pi_{s,a}^{(\theta)} \otimes \pi_{s,b}^{(\theta')}) R \right) \quad (2.14)$$

and multiplying it by an overall scalar prefactor f_{ab} ,

$$S_{ab}(\theta - \theta') = f_{ab}(\theta - \theta') \check{R}_{ab}^{(s)}(\theta - \theta'). \quad (2.15)$$

Here $\sigma_{ab} : V_a(\theta) \otimes V_b(\theta') \rightarrow V_b(\theta') \otimes V_a(\theta)$ is the permutation operator $\sigma_{ab} : v_a \otimes v_b \mapsto v_b \otimes v_a$. The prefactor $f_{ab}(\theta)$ will be constrained by the requirements of unitarity, crossing symmetry and the bootstrap principle, as we will explain in the next sections. That the right hand side of (2.15) depends only on $\theta - \theta'$, as required, follows from the fact that the universal R-matrix, as arising from Drinfeld's double construction [17], has the form $R = \sum_\gamma e_\gamma \otimes e^\gamma$ with $s(e_\gamma) = -s(e^\gamma)$ for any gradation s .

By definition, the universal R-matrix of $\widetilde{U_q(\hat{g})}$ satisfies

$$R\Delta(A) = \Delta^T(A)R \quad \forall A \in \widetilde{U_q(\hat{g})}, \quad (2.16)$$

where Δ^T is the opposite coproduct obtained by interchanging the factors of the tensor products in (2.7). The intertwining property (2.13) of $S_{ab}(\theta - \theta')$ follows immediately from (2.16) by acting with $\sigma_{ab}(\pi_a^{(\theta)} \otimes \pi_b^{(\theta')})$ on both sides.

The multi-particle S-matrix needs to similarly intertwine the action (2.10) of the symmetry on multi-particle states and from this one can deduce that the multi-particle scattering is given as the product of successive two-particle scatterings. The order in which the particles interact pairwise is irrelevant, this is the Yang-Baxter equation. Thus knowledge of the two-particle S-matrix is sufficient to describe all scattering processes.

2.3 R-matrices

The matrices $\check{R}_{ab}(\theta)$ which enter the construction (2.15) of the S-matrices have a nice structure which we want to explain here. Unfortunately the theory of quantum affine algebras has not yet been developed as far as one would wish and we therefore have to restrict our attention mainly to the untwisted algebras $U_q(g^{(1)})$. Also we know details about $\check{R}_{ab}(\theta)$ only if the irreducible $U_q(g^{(1)})$ representations π_a and π_b are both irreducible also as representations of $U_q(g_0)$ and if furthermore the decomposition of the tensor product representation $(\pi_a \otimes \pi_b)\Delta$ into irreducible representations of $U_q(g_0)$ is multiplicity free. Here g_0 denotes the finite dimensional Lie algebra associated to $g^{(1)}$. When these conditions are not satisfied, only some isolated cases of R-matrices have been determined but we hope that further developments will take place soon. The general construction of the R-matrices has been described for the case when these conditions hold in [15].

Let us start in the homogeneous gradation, i.e. the gradation with $s_0 = 1$ and $s_i = 0$ for $i = 1, \dots, r$, which we will denote by a super- or subscript h . In this gradation $\check{R}_{ab}^{(h)}(\theta) \equiv (\pi_{h,a}^{(\theta)} \otimes \pi_{h,b}^{(0)})R$ takes the form

$$\check{R}_{ab}^{(h)}(\theta) = c_{ab}(\theta) \sum_c \rho_{ab}^c(\theta) \check{P}_{ab}^c. \quad (2.17)$$

The sum runs over the same values as the sum in the decomposition $V_a \otimes V_b = \oplus_c V_c$ of the tensor product module $V_a \otimes V_b$ into irreducible $U_q(g_0)$ modules V_c . The matrix $\check{P}_{ab}^c : V_a \otimes V_b \rightarrow V_c \subset V_b \otimes V_a$ is the $U_q(g_0)$ intertwiner projecting onto V_c . $\rho_{ab}^c(\theta)$ is a function of the form

$$\rho_{ab}^c(\theta) = \prod_{l \in L_{ab}^c} \langle l \rangle, \quad \text{where } \langle l \rangle = \frac{1 - e^\theta q^l}{e^\theta - q^l}. \quad (2.18)$$

L_{ab}^c is a set of integers. For details see [15]. $c_{ab}(\theta)$ is an overall scalar prefactor. We exclude the case $q = 1$ from our analysis because at this point the $U_q(\hat{g})$ -symmetric trigonometric R-matrices collapse to rational R-matrices. Thus this case would have to be treated separately.

Now we want to transfer these results for the homogeneous gradation to a general gradation s given by the Lorentz spins of the quantum group generators. How to do this was explained in [4, section 5], but we will repeat it here in our new notation. We note that we can relate $\pi_{s,a}^{(\theta)}$ to $\pi_{h,a}^{(\mu\theta)}$ for some μ as follows:

$$\pi_{h,a}^{(\mu\theta)} \left(e^{\theta H^{(s)}} X_i^\pm e^{-\theta H^{(s)}} \right) = e^{\pm \theta \langle H^{(s)}, \alpha_i \rangle \pm \theta \mu \delta_{i0}} \pi_a(X_i^\pm) = \pi_{s,a}^{(\theta)}(X_i^\pm), \quad (2.19)$$

provided the Cartan subalgebra element $H^{(s)}$ and the constant μ satisfy

$$\langle H^{(s)}, \alpha_i \rangle + \mu \delta_{i0} = s_i, \quad i = 0, \dots, r. \quad (2.20)$$

These equations fix $H^{(s)}$ and μ . We can extract μ by multiplying with the Kac labels a_i , summing over i and using that $\sum_i a_i \alpha_i = 0$. We find that

$$\mu = \sum_{i=0}^r \frac{a_i s_i}{a_0}. \quad (2.21)$$

Relation (2.19) extends to the whole algebra

$$\pi_{s,a}^{(\theta)}(A) = \pi_{h,a}^{(\mu\theta)} \left(e^{\theta H^{(s)}} A e^{-\theta H^{(s)}} \right) \quad \forall A \in \widetilde{U_q(\hat{g})}. \quad (2.22)$$

This result allows us to relate the R-matrices

$$\begin{aligned} R_{ab}^{(s)}(\theta) &= (\pi_{s,a}^{(\theta)} \otimes \pi_{s,b}^{(0)}) R = (\pi_{h,a}^{(\mu\theta)} \otimes \pi_{h,b}^{(0)}) \left((e^{\theta H^{(s)}} \otimes 1) R (e^{-\theta H^{(s)}} \otimes 1) \right) \\ &= (\pi_a(e^{\theta H^{(s)}}) \otimes 1) R_{ab}^{(h)}(\mu\theta) (\pi_a(e^{-\theta H^{(s)}}) \otimes 1), \end{aligned} \quad (2.23)$$

and thus, finally,

$$\begin{aligned} \check{R}_{ab}^{(s)}(\theta) &= c_{ab}(\mu\theta) \sum_c \rho_{ab}^c(\mu\theta) \check{P}_{ab}^{(s)c} \\ \text{with} \quad \check{P}_{ab}^{(s)c} &= \left(1 \otimes \pi_a(e^{\theta H^{(s)}}) \right) \check{P}_{ab}^c \left(\pi_a(e^{-\theta H^{(s)}}) \otimes 1 \right). \end{aligned} \quad (2.24)$$

2.4 Unitarity and crossing symmetry

In S-matrix theory one analytically extends the S-matrix $S(\theta)$ to complex values of the rapidity. This analytic S-matrix for a relativistic quantum field theory has to satisfy certain requirements. It has to be unitary

$$S_{ba}(-\theta) \cdot S_{ab}(\theta) = 1 \otimes 1 \quad (2.25)$$

and crossing symmetric

$$S_{ab}(\theta) = (C_b^{-1} \otimes 1) (\sigma_{a\bar{b}} S_{\bar{b}a}(i\pi - \theta))^{t_1} \sigma_{a\bar{b}} (1 \otimes C_b) \quad (2.26)$$

$$= (1 \otimes C_a^{-1}) (\sigma_{\bar{a}b} S_{b\bar{a}}(i\pi - \theta))^{t_2} \sigma_{\bar{a}b} (C_a \otimes 1). \quad (2.27)$$

Here C_a is the charge conjugation matrix mapping a particle into its anti-particle. The superscript t_1 denotes transposition in the first factor, t_2 transposition in the second factor. These relations are easier to understand when seen graphically as in figure 2. See e.g. [33] for a treatment of S-matrices in 1+1 dimensions.

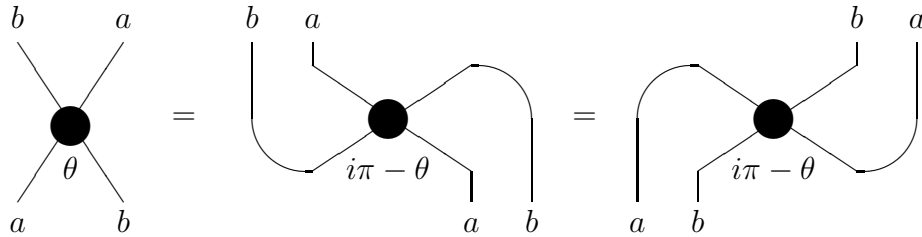


Figure 2: Crossing symmetry relations

Unitarity (2.25) can always be ensured for the S-matrix defined in (2.15). To see this we set $\theta' = 0$ and multiply (2.13) by $S_{ba}(-\theta)$ from the left

$$\begin{aligned} S_{ba}(-\theta) S_{ab}(\theta) \pi_{ab}^{(\theta\theta')}(A) &= S_{ba}(-\theta) \pi_{ba}^{(\theta'\theta)}(A) S_{ab}(\theta) \\ &= \pi_{ab}^{(\theta\theta')}(A) S_{ba}(-\theta) S_{ab}(\theta) \quad \forall A \in \widetilde{U_q(\hat{g})}. \end{aligned} \quad (2.28)$$

Because the representation $\pi_{ab}^{(\theta\theta')}$ is irreducible for generic θ, θ' , (2.28) implies by Schurr's lemma that $S_{ba}(-\theta)S_{ab}(\theta)$ is proportional to the identity. We can of course also derive this from the explicit expression (2.17) for \check{R}_{ab} and find

$$\check{R}_{ba}(-\theta)\check{R}_{ab}(\theta) = c_{ba}(-\theta)c_{ab}(\theta)(1 \otimes 1). \quad (2.29)$$

Thus $S_{ab}(\theta)$ will satisfy (2.25) if we choose f_{ab} to satisfy

$$f_{ba}(-\theta)f_{ab}(\theta) = c_{ba}^{-1}(-\theta)c_{ab}^{-1}(\theta). \quad (2.30)$$

We would like to stress that the fact that the S-matrix is unitary is not related to the question whether the corresponding field theory is unitary. Even non-unitary theories have unitary S-matrices.

Proving that $S_{ab}(\theta)$ satisfies the crossing symmetry relations (2.26) and (2.27) is a little more involved. We will need the relation between the antipode and the charge conjugation matrix. Let us first review the classical case of a Lie algebra g . To each irreducible representation π_a of g acting on a representation space V_a one defines the dual representation π_a^* acting on V_a^* by $\pi_a^*(A) = \pi_a(\gamma(A))^t \forall A \in g$, where γ is the antipode of g (which simply acts as $\gamma(A) = -A \forall A \in g$). The superscript t denotes transposition of the representation matrix. The classical charge conjugation matrix C_a^{cl} is the symmetric matrix defined by

$$\pi_a^*(A)^t \equiv \pi_a(\gamma(A)) = (C_a^{cl})^{-1} \pi_a(A)^t C_a^{cl}, \quad (2.31)$$

where $\pi_{\bar{a}}$ is the conjugate representation to π_a which is usually π_a itself but is sometimes only related to it by a diagram automorphism τ as $\pi_{\bar{a}}(A) = \pi_a(\tau(A))$.

For quantum groups $U_q(g)$ based on a finite dimensional Lie algebra g the situation is only changed slightly. The antipode \mathcal{S} of $U_q(g)$ acts as

$$\mathcal{S}(A) = q^{H_\rho} \gamma(A) q^{-H_\rho}, \quad (2.32)$$

where $\langle H_\rho, \alpha_i \rangle = d_i$ for all simple roots α_i . Thus the charge conjugation matrix is changed to

$$C_a^{fin} = C_a^{cl} \pi_a(q^{-H_\rho}). \quad (2.33)$$

For the representations $\pi_a^{(\theta)}$ of quantum affine algebras $\widetilde{U_q(\hat{g})}$ the situation is more interesting because in addition to conjugation by a charge conjugation matrix also the spectral parameter θ needs to be shifted. This is so because if we write the antipode of $U_q(\hat{g})$ as in (2.32), then H_ρ contains a component in the direction of the derivation D , i.e. the Lorentz boost generator. Alternatively we can use $\mathcal{S}(X_i^\pm) = -q^{\pm d_i} X_i^\pm$ and write

$$\pi_a^{(\theta)}(\mathcal{S}(X_i^\pm)) = -q^{\pm d_i} e^{\pm \theta s_i} \pi_a(X_i^\pm) = (e^{\pm(\theta+\xi)s_i}) \pi_a(-q^{H_\chi} X_i^\pm q^{-H_\chi}), \quad (2.34)$$

where the Cartan subalgebra element H_χ has no component in the direction of the derivation. The last equality holds if H_χ and ξ satisfy

$$e^{s_i \xi} q^{\langle H_\chi, \alpha_i \rangle} = q^{d_i} \quad (2.35)$$

If q is a real number, then this determines H_χ and ξ . We want to treat also ^{||} the case of complex q and then there is a freedom due to the $2\pi i$ ambiguity in the phases. Therefore we write $q = \exp \omega$ with ω a complex number and introduce the notation

$$[x] \equiv \text{Re}(x) + i (\text{Im}(x) \bmod 2\pi) \quad (2.36)$$

Then (2.35) is equivalent to

$$[\omega] \langle H_\chi, \alpha_i \rangle + \xi s_i = [\omega] d_i + 2\pi i m_i, \quad m_i \in \mathbb{Z} \text{ arbitrary.} \quad (2.37)$$

My realization that there is this arbitrariness can be traced back to a discussion with Nial MacKay. We are most interested in ξ . We extract it from (2.37) by multiplying with the Kac labels a_i , summing over i and using that $\sum_{i=0}^r a_i \alpha_i = 0$. We obtain

$$\xi = \frac{[\omega]}{a_0 \mu} \tilde{h} + \frac{2\pi i m}{a_0 \mu}, \quad m \in \mathbb{Z} \text{ arbitrary,} \quad (2.38)$$

where

$$\mu = \sum_{i=0}^r \frac{a_i s_i}{a_0}, \quad \tilde{h} = \sum_{i=0}^r a_i d_i. \quad (2.39)$$

We see that μ is the same as in section 2.3. \tilde{h} is the dual (k) -Coxeter number. It can be expressed in terms of the dual Coxeter number h^\vee and the twist k^\vee of the dual algebra as $\tilde{h} = k^\vee h^\vee$. We will only consider the case where $s_i = s_{\tau(i)}$. We arrive at the generalization of (2.31) to the quantum affine case

$$\pi_a^{(\theta)}(\mathcal{S}(A)) = C_a^{-1} \pi_a^{(\theta+\xi)}(A)^t C_a, \quad \forall A \in \widetilde{U_q(\hat{\mathfrak{g}})} \quad (2.40)$$

where

$$C_a = C_a^{cl} \pi_a(q^{-H_\chi}). \quad (2.41)$$

In the special case of the homogeneous gradation the shift in the spectral parameter had already been observed by Frenkel and Reshetikhin in [18]**.

We are now ready to derive the crossing relations from the following properties of the universal R-matrix

$$(\mathcal{S} \otimes 1) R = R^{-1}, \quad (1 \otimes \mathcal{S}^{-1}) R = R^{-1}. \quad (2.42)$$

We will show how to derive (2.26) from the first of these equations, the derivation of (2.27) from the second is analogous. Acting with $\pi_b^{(0)} \otimes \pi_a^{(\theta)}$ on both sides of the equation and using (2.40) and (2.29) we find

$$(C_b^{-1} \otimes 1) (R_{ba}(\xi - \theta))^{t_1} (C_b \otimes 1) = R_{ba}^{-1}(-\theta) = c_{ba}^{-1}(-\theta) c_{ab}^{-1}(\theta) \sigma_{ab} R_{ab}(\theta) \sigma_{ba}, \quad (2.43)$$

^{||}In fact we will be interested later only in the case where q is a pure phase, so that the poles of the R-matrix appear for purely imaginary θ .

**What they call the dual Coxeter number is however really the dual (k) -Coxeter number $\tilde{h} = k^\vee h^\vee$.

where we have defined $(\pi_a^{(\theta)} \otimes \pi_b^{(\theta')})R = R_{ab}(\theta - \theta')$. Using (2.15) this can be rewritten in terms of the S-matrix

$$\begin{aligned} & (C_b^{-1} \otimes 1) (\sigma_{a\bar{b}} S_{\bar{b}a}(\xi - \theta))^{t_1} \sigma_{a\bar{b}}(1 \otimes C_b) \\ &= c_{ba}^{-1}(-\theta) c_{ab}^{-1}(\theta) f_{ba}(\xi - \theta) f_{ab}^{-1}(\theta) S_{ab}(\theta). \end{aligned} \quad (2.44)$$

This produces (2.26) if

$$\xi = i\pi \quad \text{and} \quad f_{ab}(i\pi - \theta) = f_{ab}^{-2}(\theta) f_{ba}(-\theta), \quad (2.45)$$

thus putting further strong constraints on the scalar prefactor f_{ab} . We observe that crossing symmetry places a constraint on the possible gradations

$$i\pi = \xi = \frac{[\omega]}{a_0\mu} \tilde{h} + \frac{2\pi im}{a_0\mu}, \quad m \in \mathbb{Z} \text{ arbitrary.} \quad (2.46)$$

Thus only gradations (s) for which

$$a_0\mu \equiv \sum_{i=0}^r a_i s_i = \frac{[\omega]}{i\pi} \tilde{h} + 2m, \quad m \in \mathbb{Z} \quad (2.47)$$

can lead to crossing symmetric S-matrices.

2.5 S-matrix poles

A lot of information is contained in the pole structure of the analytic S-matrix. Indeed, the whole mass spectrum and the three-particle fusion rules can be read off from the location of the simple poles. Conversely, knowledge of the spectrum and the three-particle couplings determines the pole structure of the S-matrices. Because the S-matrices of integrable quantum field theories have to also obey other constraints, in particular the bootstrap principle, only very particular kinds of spectra can be realized in these theories.

If particles of type a and b can form a bound state of type c with mass

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c, \quad (2.48)$$

then this usually leads to a simple pole of $S_{ab}(\theta)$ at $\theta = iu_{ab}^c$ corresponding to the propagation of a particle c in the direct channel, as depicted in figure 3 a). By crossing symmetry it also leads to poles in $S_{b\bar{a}}$ and $S_{\bar{b}a}$ at $\theta = i\bar{u}_{ab}^c = i(\pi - u_{ab}^c)$, see figure 3 b) and c).

Higher order poles in the S-matrix are due to higher order processes like the one depicted in figure 3 d). When and where such processes give higher order poles according to the Coleman-Thun mechanism is well explained in [5, 6].

We say that a bound state with a mass m_c given by (2.48) only “usually” leads to a simple pole at $\theta = iu_{ab}^c$ because it has been observed [11, 12] that higher order

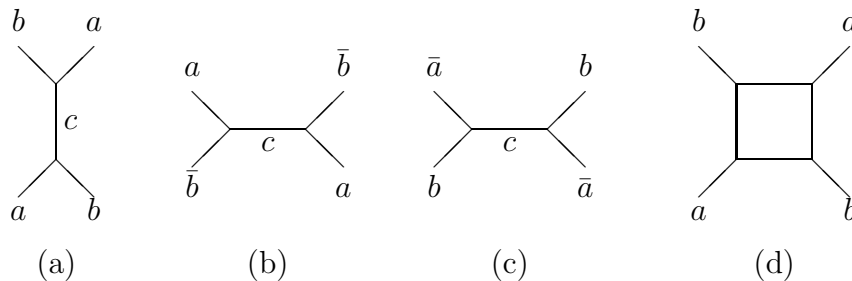


Figure 3: Processes leading to poles in the S-matrix

processes like that in figure 3 d) might take place at a value of the rapidity which is very close to iu_{ab}^c . In that case the outcome may be a simple pole slightly shifted away from iu_{ab}^c . This phenomenon is important in affine Toda theory.

The residue of a particle pole at $\theta = iu_{ab}^c$ of the S-matrix $S_{ab}(\theta) : V_a \otimes V_b \rightarrow V_b \otimes V_a$ should project onto an irreducible submodule V_c , i.e., $\text{Res}(S_{ab}(\theta = iu_{ab}^c)) : V_a \otimes V_b \rightarrow V_c \subset V_b \otimes V_a$. This is just saying that the scattering process at the pole is dominated by the propagation of particles of type c only. Our S-matrix $S_{ab}(\theta) = f_{ab}(\theta)\check{R}_{ab}^{(s)}(\theta)$ has the potential of producing such a phenomenon. Looking at (2.24) we see that at any value of θ at which some of the $\rho_{ab}^d(\mu\theta)$ have a pole, the residue of $\check{R}_{ab}^{(s)}(\theta)$ indeed projects onto a submodule. Unless the scalar prefactor had a zero at the same location, the particles a and b would form a bound state transforming in the corresponding sub-representation. If only one of the ρ_{ab}^d , let us say ρ_{ab}^c , has a pole at $\theta = iu_{ab}^c$ then the bound state particle transforms in the irreducible representation π_c of $U_q(g)$. In general however, several of the ρ_{ab}^d , let us say $\rho_{ab}^{c_1}, \dots, \rho_{ab}^{c_n}$, will have a pole, in which case the bound state particle transforms as the representation $\pi_c = \pi_{c_1} \oplus \dots \oplus \pi_{c_n}$ which is reducible as a representation of $U_q(g)$ but still irreducible as a representation of $U_q(\hat{g})$.

If a theory has a particle spectrum containing the multiplets a_1, \dots, a_n , corresponding to some irreducible $U_q(\hat{g})$ modules V_{a_1}, \dots, V_{a_n} , then one has to ensure that the corresponding S-matrices $S_{a_i a_j}$ have particle poles only at locations such that the residue projects onto one of the modules V_{a_1}, \dots, V_{a_n} . If there was a simple pole with a residue projecting onto some other module V_b then a corresponding particle multiplet b would also have to be in the spectrum of the theory. If there was a simple pole at a location where the residue is not a projector onto a submodule, then that pole must be checked to have an explanation in terms of the Coleman-Thun mechanism. For examples of such poles see [12],[9].

We realize from these comments that one should choose the prefactor f_{ab} in such a way that it cancels many of the poles in the expression (2.17) for the \check{R} -matrix which would lead to bound states which do not actually exist in the theory. That doing this is a difficult task is due to the bootstrap principle.

The bootstrap principle of S-matrix theory states that if $S_{ab}(\theta)$ has a simple pole corresponding to a particle of type c , then the S-matrices S_{dc} describing the scattering of a particle type c with any other particle type d are expressed in terms

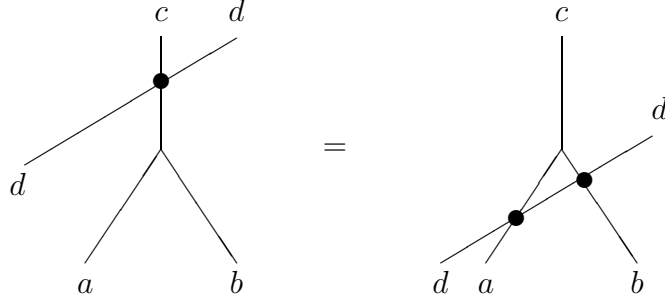


Figure 4: The bootstrap principle

of the S-matrices S_{ad} and S_{bd} . This is expressed pictorially in figure 4 and through the formula

$$S_{dc}(\theta) (1 \otimes P_{ab}^c) = (1 \otimes S_{db}(\theta + i\bar{u}_{bc}^a)) (S_{da}(\theta - i\bar{u}_{ac}^b) \otimes 1), \quad (2.49)$$

where P_{ab}^c is the projector onto V_c in $V_a \otimes V_b$. That the matrices $\check{R}_{ab}(\theta)$ satisfy such a relation follows from the defining property of the universal R-matrix [17]

$$(1 \otimes \Delta)R = R_{13}R_{12}. \quad (2.50)$$

That also $S_{ab}(\theta)$ satisfies (2.49) puts further constraints on the scalar prefactor f_{ab} and in particular on the location of its poles and zeros.

2.6 General remarks on the quantum spectrum

The simple particle poles of the S-matrix will be at locations at which the R-matrix projects onto a submodule. This implies that they occur at values of θ at which $e^{\mu\theta} = q^l$, where l is one of the numbers in eq. (2.18). Writing again $q = e^\omega$ we see that the potential particle poles occur at

$$\theta = l \frac{[\omega]}{\mu} + \frac{2\pi ip}{\mu}, \quad p \in \mathbb{Z}. \quad (2.51)$$

At which of these potential locations the S-matrix will really have poles is of course determined by the zeros of the scalar prefactor. Clearly we will have to have ω purely imaginary (i.e. q a pure phase) in order for the poles to lie on the imaginary axis, as is required for stable particle poles. μ is real by construction because the Lorentz spins of the symmetry charges are real.

Let us assume that we know the classical spectrum of particles and their coupling rules in the integrable field theory under study. Then we know the classical locations of the poles. We can identify them among the poles in (2.51) and then we can read off their dependence of \hbar and the coupling constant from the dependence of $[\omega]$ and μ on these. By this procedure we can derive the full quantum mass ratios of the particles. (When we say “particle” we mean of course not only fundamental particles but also solitons, breathers, excited solitons etc.).

This is of great significance, because it is usually next to impossible to calculate the quantum corrections to masses to all orders. Usually, even calculating just the first order correction to the masses is a formidable task, as evidenced by recent calculations of the first mass corrections to the soliton masses in affine Toda theory [21, 16, 27]. On the other hand it is usually simpler to determine the existence of symmetry algebras to all orders, because here one can often make use of the fact that no further anomalies can appear beyond a certain orders in perturbation theory. See for example the proof of quantum integrability of real coupling Toda theory [13]. Similarly Bernard and LeClair have argued the quantum affine symmetry in imaginary coupling Toda theory to all orders by a scaling argument [3].

From the freedom of choosing the integer p in (2.51) we see that to any particle transforming in a particular representation c there can be further particle states transforming in the same representation, corresponding to other values of p . These could be interpreted as excitations of the particle. Because particle poles have to lie on the physical sheet, i.e. at $0 < \text{Im}\theta < i\pi$, these states can exist only for the integers p in a certain range. This range is determined by the gradation through the parameter μ . Conversely, if one does not know the gradation, but knows the tower of excited states in the spectrum, then one can deduce μ by the separation between these states from (2.51).

It is illuminating to rewrite the pole locations in (2.51) using the constraint (2.47) coming from crossing symmetry. We obtain

$$\theta = la_0 \frac{i\pi}{\tilde{H}} + \frac{2\pi ip}{\mu}. \quad (2.52)$$

where we have introduced a sort of “quantum” dual (k)-Coxeter number

$$\tilde{H} = \tilde{h} + \frac{2\pi im}{[\omega]} \quad (2.53)$$

This is very reminiscent of the “quantum” Coxeter number H which appears in the pole locations of the scalar S-matrices for the fundamental particles in real coupling affine Toda theory [11, 12].

3 $U_q(c_n^{(1)})$ symmetric S-matrices

In this section we will give a concrete example for a consistent set of S-matrices for a theory with the quantum affine symmetry based on the Kac-Moody algebra $c_n^{(1)}$.

To specify a theory we have to not only give the symmetry algebra but also state which representations of the symmetry algebra will occur as particle multiplets. In this example we choose to include particles transforming in all the fundamental representations of $U_q(\widetilde{c_n^{(1)}})$. The fundamental representations are the representations whose highest weight is a fundamental weight λ_a of c_n . The fundamental weights are defined by the property that $\lambda_a \cdot \alpha_b^\vee = \delta_{ab}$, $a, b = 1, \dots, r$.

We have two related reasons for choosing this particular example.

1) There is a lagrangian field theory which exhibits exactly these features: $d_{n+1}^{(2)}$ Toda theory at imaginary coupling is believed to have a $c_n^{(1)}$ quantum affine symmetry [3] and its solitons species [31] correspond to the fundamental representations. It is a long standing problem to construct the corresponding soliton S-matrices and it is hoped that this example will lead to the solution of that problem.

2) Hollowood [23] has already attempted to construct S-matrices of this type. He finds it to be impossible to give a suitable scalar prefactor to implement the correct pole structure. We can now see that this failure is due to the fact that he implicitly worked with a gradation with $\mu = [\omega]\tilde{h}$. Using a more generic gradation, the construction becomes possible.

3.1 The R-matrices

For $U_q(c_n^{(1)})$ the spectral decomposition of all the R-matrices \check{R}_{ab} , with π_a and π_b being any two fundamental representations, are known. In the homogeneous gradation they were given in [23], see also [26, 15]. Using the same notation as in (2.17),(2.18), they are

$$\check{R}_{ab}^{(h)}(\theta) = c_{ab}(\theta) \sum_{c=0}^{\min(b,n-a)} \sum_{d=0}^{b-c} \prod_{i=1}^c \langle a-b+2i \rangle \prod_{j=1}^d \langle 2n+2-a-b+2j \rangle \check{P}_{ab}^{(cd)}, \quad (3.1)$$

where by (cd) we denote the irreducible $U_q(c_n)$ representation with highest weight $\lambda_{a+c-d} + \lambda_{b-c-d}$. Without loss of generality we have chosen $a \geq b$. This rather complicated formula is encoded in the “extended tensor product graph” displayed in figure 5. Each node in that graph corresponds to an irreducible $U_q(c_n)$ representation which appears in the tensor product of the two fundamental representations a and b . Thus they correspond to the intertwining projectors \check{P} in (3.1). The prefactor of a particular \check{P} is obtained as product of $\langle l \rangle$ factors, one for each link on a path from the corresponding node on the graph to the top node. It turns out that the choice of path does not matter. The integer l in the $\langle l \rangle$ factor corresponding to a particular link is half the difference between the values of the Casimirs of the connected nodes. The details of this construction in the general case are described in [15].

3.2 The scalar prefactors

Hollowood [23], has found a prefactor $g_{ab}(\theta)$ such that the matrix

$$\tilde{S}_{ab}^{(h)}(\theta) = g_{ab}(\theta) \check{R}_{ab}^{(h)}(\theta) \quad (3.2)$$

satisfies the unitarity relation (2.25) and the crossing relations (2.26),(2.27) and, in addition, has no poles on the physical strip. For the details of this prefactor we refer the reader to Hollowood’s paper [23].

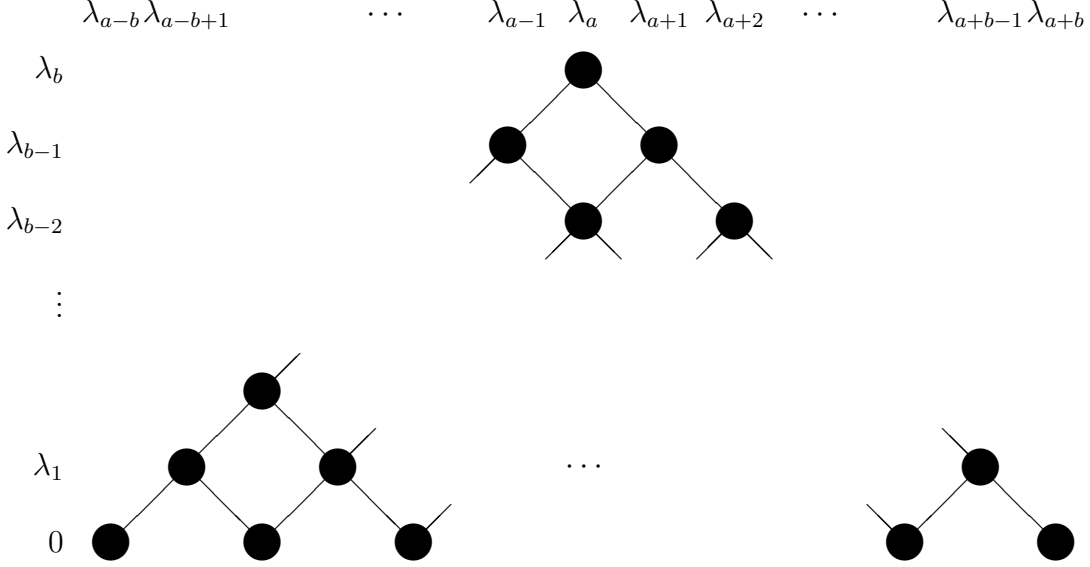


Figure 5: The extended tensor product graph for the product $V(\lambda_a) \otimes V(\lambda_b)$ ($a \geq b$) of two arbitrary fundamental representations of C_n . The nodes correspond to representations whose highest weight is given by the sum of the weight labeling the column and the weight labeling the row. If $a + b > n$ then the graph extends to the right only up to λ_n .

$\tilde{S}_{ab}^{(h)}(\theta)$ is in the homogeneous gradation, we need to transform it to the gradation (s) determined by the Lorentz spins \tilde{s}_i of the quantum group charges. We have learned in section 2.3 how to achieve this.

$$\tilde{S}_{ab}^{(s)}(\theta) = \left(\pi_a(e^{\theta H^{(s)}}) \otimes 1 \right) \tilde{S}_{ab}^{(h)}(\mu\theta) \left(1 \otimes \pi_a(e^{-\theta H^{(s)}}) \right). \quad (3.3)$$

Because of the absence of any particle poles, this matrix can not yet be an S-matrix. Rather it needs to be multiplied by another prefactor $X_{ab}(\theta)$

$$S_{ab}(\theta) = X_{ab}(\theta) \tilde{S}_{ab}^{(s)}(z), \quad (3.4)$$

$X_{ab}(\theta)$ has to satisfy the S-matrix axioms by itself and has to have particle poles at those values of θ for which the R-matrix $R_{ab}^{(s)}(\theta)$ projects onto subrepresentations corresponding to another fundamental representation c . We claim that such a prefactor is given by

$$X_{ab}(\theta) = \prod_{p=1}^b \{a - b - 1 + 2p\}_{\tilde{H}} \{\tilde{H} - a + b + 1 - 2p\}_{\tilde{H}}, \quad (3.5)$$

where we use the notation

$$\{a\}_{\tilde{H}} = \frac{(a-1)_{\tilde{H}}(a+1)_{\tilde{H}}}{(a-1+B)_{\tilde{H}}(a+1-B)_{\tilde{H}}}, \quad (a)_{\tilde{H}} = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{2\tilde{H}}a\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{2\tilde{H}}a\right)}. \quad (3.6)$$

$\tilde{H} = \tilde{h} - B$ and B is a parameter which we will be related to q and μ . For $c_n^{(1)}$ the dual (k) -Coxeter number is $\tilde{h} = 2n + 2$.

These factors X_{ab} are nothing else but the scalar S-matrices of the fundamental particles of $d_{n+1}^{(2)}$ Toda theory which were found in [12]. In that reference the pole structure of these X_{ab} for $0 < B < 2$ has been investigated and all poles on the physical strip have been shown to either be particle poles or to arise from the Coleman-Thun mechanism. In particular some simple poles are shifted away from their single-particle position by higher order processes. The remaining true particle poles were used to check consistency with the bootstrap principle.

Thus the only thing which remains to be checked is that at the particle poles of $X_{ab}(\theta)$, $\tilde{S}_{ab}^{(s)}(\theta)$ projects onto submodules. If $a + b \leq n$, $X_{ab}(\theta)$ has a particle pole at

$$\theta_{\text{pole}} = i\pi \frac{a+b}{\tilde{H}}, \quad (3.7)$$

We read off from (3.1) that the residue of $\check{R}_{ab}^{(h)}(\theta)$ at $\theta = [\omega](a+b)$ projects onto V_{a+b} . Correspondingly, $\tilde{S}_{ab}^{(s)}$ projects onto V_{a+b} at $\theta = \frac{[\omega]}{\mu}(a+b)$. Thus if we set

$$\tilde{H} = i\pi \frac{\mu}{[\omega]} \quad (3.8)$$

then indeed the particle pole in X_{ab} corresponds to the propagation of a particle of type $a+b$. Using the constraint (2.47) coming from crossing symmetry we see that this implies that \tilde{H} is related to the dual (k) -Coxeter number \tilde{h} by

$$\tilde{H} = \tilde{h} + \frac{2\pi im}{[\omega]}, \quad \text{i.e., } B = -\frac{2\pi im}{[\omega]}. \quad (3.9)$$

From the location of the pole we can calculate the quantum masses of the particles up to an overall scale M by using formula (2.48). We find

$$M_a = M \sin\left(\frac{a\pi}{\tilde{H}}\right). \quad (3.10)$$

The simple poles in X_{ab} at $\theta = i\pi \frac{\tilde{H}-a-b}{\tilde{H}}$ are due to the propagation of the same particle type $a+b$ but in the crossed channel. The simple poles in X_{ab} with $a+b > n$ at $\theta = i\pi \frac{a+b}{\tilde{H}}$ are not particle poles. Rather the single particle poles are shifted by higher order processes as been explained in [15].

The reason why Hollowood in [23] was not able to find a consistent S-matrix was that he was implicitly working with a gradation which corresponds to $\tilde{H} = \tilde{h}$ and at this particular point the prefactors X_{ab} which we have found reduce to 1.

3.3 The breathers

There is one more set of simple poles which we have not discussed yet and which lie on the physical strip only if B is negative. In the $d_{n+1}^{(2)}$ Toda theory at real

coupling constant, B is positive and thus these poles have not appeared in [15]. If B is negative these simple poles lead to more states in the spectrum of the $U_q(c_n^{(1)})$ symmetric theory and in analogy to affine Toda theory we will call these bound states “breathers”.

$X_{aa}(\theta)$ has a single pole at

$$\theta_{\text{pole}} = i\pi \frac{\tilde{h}}{\tilde{H}}. \quad (3.11)$$

At this value of θ , $\tilde{S}_{ab}^{(s)}$ projects onto the trivial one-dimensional representation V_0 , as can be seen from (3.1). Thus two solitons of type a can create a breather singlet state of mass

$$\left(M_a^{\text{breather}}\right)^2 = 2M_a^2 \left(1 + \cos\left(\pi \frac{\tilde{h}}{\tilde{H}}\right)\right). \quad (3.12)$$

The pole in $X_{aa}(\theta)$ at $\theta = i\pi \frac{-B}{\tilde{H}}$ is due to the propagation of the same breather state in the crossed channel. The \tilde{S} -matrices describing the scattering of these breathers could be obtained from the S_{ab} by applying the bootstrap.

More poles will occur for $|B| > 1$, leading to further excited particle states

4 Conclusion

Quantum affine algebras have been shown to be a practical tool to construct exact 1+1 dimensional relativistic S-matrices which satisfy all the axioms of S-matrix theory. This has been explicitly demonstrated by an example.

By a careful study of the consequences of the Lorentz spins of the symmetry charges and the requirement of crossing symmetry we have found the formula (2.52) for the location of the particle poles which eventually determine the quantum masses. It is pleasant to see a certain “quantum” dual Coxeter number to appear in this formula, mirroring the way in which a “quantum” Coxeter number appeared in the pole locations of the fundamental particles of affine Toda theory.

We expect that these S-matrices will find applications in several 1+1 dimensional quantum field theories, in particular as the soliton S-matrices of quantum affine Toda theory. This will allow the further study in these theories of such properties as the strong-coupling — weak-coupling duality, the breather — particle duality and the algebra — dual algebra duality.

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